Pensieve header: Analysis of k=2 invariants in QU.

```
In[•]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\SL2Portfolio"];
        << KnotTheory`
        << "SL2PortfolioProgram.m"
```

```
Loading KnotTheory` version of January 20, 2015, 10:42:19.1122.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
In[•]:= OverbayP2Data = Get["C:\\drorbn\\AcademicPensieve\\People\\Overbay\\OverbayP2Data.m"];
        OP2[K_Knot] := K /. OverbayP2Data /. T → T^(1/2);
```

```
In[•]:= Last[OverbayP2Data]
```

```
Out[•]= Knot[8, 21] → 2670 + 1/T^14 + 4/T^12 - 60/T^10 + 276/T^8 - 775/T^6 +
        1550/T^4 - 2331/T^2 - 2331 T^2 + 1550 T^4 - 775 T^6 + 276 T^8 - 60 T^10 + 4 T^12 + T^14
```

```
In[•]:= $p = 5; $k = 2; $U = QU;
```

```
In[•]:= SCΘ[p_] := Collect[ CΘ@@𝒪CU[{y, a, x}, p] /. {CU → Times, γ | ħ → 1}, ϵ, Simplify];
        SQΘ[p_] := Collect[ QΘ@@𝒪QU[{y, a, x}, p] /. {QU → Times, γ | ħ → 1}, ϵ, Simplify];
```

```
In[•]:= 𝔼[L_, Q_, P_]$k_ := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
        𝔼d_→r_[L_, Q_, P_]$k_ := 𝔼d→r@@𝔼[L, Q, P]$k;
        𝔼3@𝔼[ω_, L_, Q_, Ps_] := CF /@ 𝔼[L, ω^-1 Q, ω^-1 (ω^-4 ϵ)^(-1+Range@Length@Ps .Ps)]$k;
        𝔼4@𝔼[L_, Q_, P_] := Module[
            {ω = Normal[P]^-1 /. ϵ → 0, Ps = CoefficientList[P, ϵ]},
            CF /@ 𝔼[ω, L, ω Q, ω^(-3+4 Range@Length@Ps Ps)]];
        𝔼3@𝔼sp___[as___] := 𝔼3@𝔼[as] /. 𝔼 → 𝔼sp;
        𝔼4@𝔼sp___[as___] := 𝔼4@𝔼[as] /. 𝔼 → 𝔼sp;
```

```
In[•]:= Clear[QP, ω];
        QP[Knot[n_, k_]] := QP[Knot[n, k]] = Collect[Module[{fname},
              fname = "../SL2Invariant/k=2/Data/" <> ToString[n] <> "_" <> ToString[k] <> ".m";
              Collect[𝔼3[Get[fname]⟦2, 2⟧]⟦3⟧ // Normal, ϵ, Simplify]
            ], ϵ, CF];
        ω[K_Knot] := ω[K] = Factor[(QP@K /. ϵ → 0)^-1];
        LQP[K_] := Collect[Normal@Log[ω[K] QP[K] + O[ϵ]^3], ϵ, Simplify];
        c_k_,d_[K_Knot] :=
         Factor[SeriesCoefficient[LQP[K], {y, 0, 0}, {ϵ, 0, k}, {a, 0, d}] ω[K]^(1+k)]
```

```
In[ ]:= H[p_] := If[TrueQ@Simplify[p == (p /. T → 1/T)],
           σ@@CoefficientList[Expand@Together[p] /. T^n_ /; n < 0 :↦ 0, T], p];
        H[p_] := If[TrueQ@Simplify[p == (p /. T → 1/T)],
           Style[Expand@Together[p] /. T^n_ /; n < 0 :↦ 0, Background → Yellow], p];
        H[p_] := If[TrueQ@Simplify[p == (p /. T → 1/T)], Style[p, Background → Yellow], p];
```

```
In[ ]:= Total[Table[
          Simplify[(c0,0[K] == 0 ∧ c1,1[K] == 2 ω[K] T ∂T ω[K])],
          {K, AllKnots[{3, 8}]}]]
```

```
Out[ ]= 35 True
```

```
In[ ]:= MatrixForm[Table[
          H /@ Factor /@ {ω[K], c1,1[K]/(T - T^-1), p1 = (T (-c1,0[K] + ω[K] T ∂T ω[K]))/(T - 1)^2, 2 ω[K] c2,0[K] -
              ω[K] c2,1[K]/2, 2 ω[K] c2,0[K], ω[K] c2,1[K]/2, c2,2[K]/ω[K], OP2[K]},
          {K, AllKnots[{3, 7}]}
        ]]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix}
\frac{1-T+T^2}{T} & \frac{2(1-T+T^2)}{T} & \frac{1+T^2}{T} & \\
-\frac{1-3T+T^2}{T} & \frac{2(1-3T+T^2)}{T} & 0 & \\
\frac{1-T+T^2-T^3+T^4}{T^2} & \frac{2(2-T+2T^2)(1-T+T^2-T^3+T^4)}{T^3} & \frac{(1+T^2)(2+T^2+2T^4)}{T^3} & \\
\frac{2-3T+2T^2}{T} & \frac{4(2-3T+2T^2)}{T} & \frac{5-4T+5T^2}{T} & \\
-\frac{(-2+T)(-1+2T)}{T} & \frac{4(-2+T)(-1+2T)}{T} & \frac{1-4T+T^2}{T} & \\
-\frac{1-3T+3T^2-3T^3+T^4}{T^2} & \frac{2(2-3T+2T^2)(1-3T+3T^2-3T^3+T^4)}{T^3} & \frac{1-4T+4T^2-4T^3+4T^4-4T^5+T^6}{T^3} & -4 \\
\frac{1-3T+5T^2-3T^3+T^4}{T^2} & \frac{2(2-3T+2T^2)(1-3T+5T^2-3T^3+T^4)}{T^3} & 0 & \\
\frac{1-T+T^2-T^3+T^4-T^5+T^6}{T^3} & \frac{2(3-2T+4T^2-2T^3+3T^4)(1-T+T^2-T^3+T^4-T^5+T^6)}{T^5} & \frac{(1+T^2)(3+2T^2+4T^4+2T^6+3T^8)}{T^5} & -3+12T-36T^2+84T^3-174T^4+24 \\
\frac{3-5T+3T^2}{T} & \frac{6(3-5T+3T^2)}{T} & \frac{2(7-8T+7T^2)}{T} & \\
\frac{2-3T+3T^2-3T^3+2T^4}{T^2} & \frac{2(4-3T+4T^2)(2-3T+3T^2-3T^3+2T^4)}{T^3} & -\frac{9-8T+16T^2-12T^3+16T^4-8T^5+9T^6}{T^3} & 1-8T+45T^2-16 \\
\frac{4-7T+4T^2}{T} & \frac{8(4-7T+4T^2)}{T} & -\frac{8(3-4T+3T^2)}{T} & \\
\frac{2-4T+5T^2-4T^3+2T^4}{T^2} & \frac{8(1-T+T^2)(2-4T+5T^2-4T^3+2T^4)}{T^3} & \frac{9-16T+29T^2-28T^3+29T^4-16T^5+9T^6}{T^3} & 1+4T-92T^2+428T^3- \\
-\frac{1-5T+7T^2-5T^3+T^4}{T^2} & \frac{2(-2+T)(-1+2T)(1-5T+7T^2-5T^3+T^4)}{T^3} & \frac{1-8T+19T^2-20T^3+19T^4-8T^5+T^6}{T^3} & -3+44 \\
\frac{1-5T+9T^2-5T^3+T^4}{T^2} & \frac{2(-2+T)(-1+2T)(1-5T+9T^2-5T^3+T^4)}{T^3} & -\frac{3-8T+3T^2}{T} & -5+52
\end{pmatrix}$$

In[•]:=
```
MatrixForm[mat = Table[
    {q1 = ω[K], q2 = T (-c1,0[K] + ω[K] T ∂T ω[K]) / (T - 1)^2, (* T∂T ω[K] / (T - T^-1),
     T∂T (T∂T ω[K]),*) q3 = -2 c2,0[K] + ω[K] c2,1[K], q4 = ω[K] (c2,1[K] - ω[K] c2,2[K]) / (-2 T + 2/T),
     q5 = ω[K]^2 c2,2[K] / 2, q6 = ω[K]^2 T (-c1,0[K] + ω[K] T ∂T ω[K]) / (T - 1)^2};
    Times @@@ Subsets[Together /@ {q1, q2, q3, q4, q5, q6}, 5] /. T → -1,
    {K, AllKnots[{3, 9}]}
   ]];
Dimensions[mat]
MatrixRank[mat]
```

Out[•]= {84, 63}

Out[•]= 63

In[•]:=
```
MatrixForm[mat = Table[
    {q1, q2, q3, q4, q5} =
     Together /@ {ω[K], T (-c1,0[K] + ω[K] T ∂T ω[K]) / (T - 1)^2, (* T∂T ω[K] / (T - T^-1), T∂T (T∂T ω[K]),*)
       -2 c2,0[K] + ω[K] c2,1[K], ω[K] (c2,1[K] - ω[K] c2,2[K]) / (-2 T + 2/T), ω[K]^2 c2,2[K] / 2};
    Join @@ Table[q1^k {q2, q3, q4, q5}, {k, 0, 5}] /. T → -1,
    {K, AllKnots[{3, 8}]}
   ]];
Dimensions[mat]
MatrixRank[mat]
```

Out[•]= {35, 24}

Out[•]= 24

In[•]:=
```
NullSpace[mat].Join @@ Table[qq1^k {qq2, qq3, qq4, qq5}, {k, 0, 5}]
```

Dot: Tensors {} and

{qq2, qq3, qq4, qq5, qq1 qq2, qq1 qq3, qq1 qq4, qq1 qq5, qq1^2 qq2, qq1^2 qq3, qq1^2 qq4, qq1^2 qq5, qq1^3 qq2, qq1^3 qq3, qq1^3 qq4, qq1^3 qq5, qq1^4 qq2, qq1^4 qq3, qq1^4 qq4, qq1^4 qq5, qq1^5 qq2, qq1^5 qq3, qq1^5 qq4, qq1^5 qq5} have incompatible shapes.

Out[•]= {}.{qq2, qq3, qq4, qq5, qq1 qq2, qq1 qq3, qq1 qq4, qq1 qq5, qq1^2 qq2, qq1^2 qq3, qq1^2 qq4, qq1^2 qq5, qq1^3 qq2, qq1^3 qq3, qq1^3 qq4, qq1^3 qq5, qq1^4 qq2, qq1^4 qq3, qq1^4 qq4, qq1^4 qq5, qq1^5 qq2, qq1^5 qq3, qq1^5 qq4, qq1^5 qq5}

```
In[•]:= Most /@ (Join @@ (Permutations /@ (PadRight[#, 6] & /@ IntegerPartitions[3])))
```

```
Out[•]= {{3, 0, 0, 0, 0}, {0, 3, 0, 0, 0}, {0, 0, 3, 0, 0}, {0, 0, 0, 3, 0},
  {0, 0, 0, 0, 3}, {0, 0, 0, 0, 0}, {2, 1, 0, 0, 0}, {2, 0, 1, 0, 0},
  {2, 0, 0, 1, 0}, {2, 0, 0, 0, 1}, {2, 0, 0, 0, 0}, {1, 2, 0, 0, 0},
  {1, 0, 2, 0, 0}, {1, 0, 0, 2, 0}, {1, 0, 0, 0, 2}, {1, 0, 0, 0, 0},
  {0, 2, 1, 0, 0}, {0, 2, 0, 1, 0}, {0, 2, 0, 0, 1}, {0, 2, 0, 0, 0}, {0, 1, 2, 0, 0},
  {0, 1, 0, 2, 0}, {0, 1, 0, 0, 2}, {0, 1, 0, 0, 0}, {0, 0, 2, 1, 0}, {0, 0, 2, 0, 1},
  {0, 0, 2, 0, 0}, {0, 0, 1, 2, 0}, {0, 0, 1, 0, 2}, {0, 0, 1, 0, 0}, {0, 0, 0, 2, 1},
  {0, 0, 0, 2, 0}, {0, 0, 0, 1, 2}, {0, 0, 0, 1, 0}, {0, 0, 0, 0, 2}, {0, 0, 0, 0, 1},
  {1, 1, 1, 0, 0}, {1, 1, 0, 1, 0}, {1, 1, 0, 0, 1}, {1, 1, 0, 0, 0}, {1, 0, 1, 1, 0},
  {1, 0, 1, 0, 1}, {1, 0, 1, 0, 0}, {1, 0, 0, 1, 1}, {1, 0, 0, 1, 0}, {1, 0, 0, 0, 1},
  {0, 1, 1, 1, 0}, {0, 1, 1, 0, 1}, {0, 1, 1, 0, 0}, {0, 1, 0, 1, 1}, {0, 1, 0, 1, 0},
  {0, 1, 0, 0, 1}, {0, 0, 1, 1, 1}, {0, 0, 1, 1, 0}, {0, 0, 1, 0, 1}, {0, 0, 0, 1, 1}}
```

```
In[•]:= MatrixForm[mat = Table[
          vs = (Together /@ {ω[K], p1 = (T (-c1,0[K] + ω[K] T ∂T ω[K]))/(T - 1)^2,
              (T ∂T p1)/(T - T^-1), (T ∂T ω[K])/(T - T^-1), T ∂T (T ∂T ω[K]), -2 c2,0[K] + ω[K] c2,1[K],
              ω[K] (c2,1[K] - ω[K] c2,2[K])/(-2 T + 2/T), ω[K]^2 c2,2[K]/2 (*,OP2[K]*)}) /. T → 22/7;
          PS = Select[Most /@ (Join @@ (Permutations /@ (PadRight[#, 1 + Length[vs]] & /@
                IntegerPartitions[4]))), (Total[#[[-3 ;; -1]]] == 1 ∧ #[[3]] + #[[5]] ≤ 1) &];
          Table[Times @@ (v /@ Range@Length@vs)^ps, {ps, PS}] /. v[i_] :> vs[[i]],
          {K, AllKnots[{3, 9}]}
        ]];
      Dimensions[mat]
      MatrixRank[mat]
```

```
Out[•]= {84, 120}
```

```
Out[•]= 84
```

In[•]:=
```
NullSpace[mat].Table[Times @@ (v /@ Range@Length@vs)^ps, {ps, PS}]
```

Out[•]=
```
{0.235532 v[5] - 0.468453 v[1] v[5] + 0.232878 v[1]^2 v[5] -
  0.000483308 v[2] v[5] + 0.000475733 v[1] v[2] v[5] + 2.30688 × 10^-7 v[2]^2 v[5] +
  0.179815 v[3] v[5] - 0.178798 v[1] v[3] v[5] - 0.000175428 v[2] v[3] v[5] +
  0.0343027 v[3]^2 v[5] - 0.0150066 v[4] v[5] + 0.0149227 v[1] v[4] v[5] +
  0.0000146863 v[2] v[4] v[5] - 0.00573045 v[3] v[4] v[5] + 0.000239995 v[4]^2 v[5] +
  0.0000404166 v[5] v[6] - 0.0000495274 v[1] v[5] v[6] - 1.17055 × 10^-8 v[2] v[5] v[6] +
  0.0000201261 v[3] v[5] v[6] - 1.79393 × 10^-6 v[4] v[5] v[6] + 4.14679 × 10^-10 v[5] v[6]^2 +
  0.286877 v[7] - 0.577451 v[1] v[7] + 0.290504 v[1]^2 v[7] + 0.000305026 v[2] v[7] -
  0.000294998 v[1] v[2] v[7] - 1.59076 × 10^-7 v[2]^2 v[7] + 0.225099 v[3] v[7] -
  0.22638 v[1] v[3] v[7] + 0.000106165 v[2] v[3] v[7] + 0.0440797 v[3]^2 v[7] -
  0.0202173 v[4] v[7] + 0.0203202 v[1] v[4] v[7] - 8.7362 × 10^-6 v[2] v[4] v[7] -
  0.00789966 v[3] v[4] v[7] + 0.00035229 v[4]^2 v[7] - 0.0000295223 v[6] v[7] +
  0.0000361118 v[1] v[6] v[7] + 1.22321 × 10^-8 v[2] v[6] v[7] -
  0.000014402 v[3] v[6] v[7] + 1.24315 × 10^-6 v[4] v[6] v[7] - 4.22287 × 10^-10 v[6]^2 v[7],
 0.290331 v[5] - 0.579322 v[1] v[5] + 0.288941 v[1]^2 v[5] + 0.0000547222 v[2] v[5] -
  0.0000531463 v[1] v[2] v[5] - 5.73002 × 10^-8 v[2]^2 v[5] + 0.222614 v[3] v[5] -
  0.22206 v[1] v[3] v[5] + 0.0000187087 v[2] v[3] v[5] + 0.0426746 v[3]^2 v[5] -
  0.0191361 v[4] v[5] + 0.0190876 v[1] v[4] v[5] - 1.63149 × 10^-6 v[2] v[4] v[5] -
  0.007336 v[3] v[4] v[5] + 0.000315219 v[4]^2 v[5] - 0.0000473907 v[5] v[6] +
  0.000051275 v[1] v[5] v[6] - 1.48271 × 10^-9 v[2] v[5] v[6] - 0.0000203981 v[3] v[5] v[6] +
  1.82607 × 10^-6 v[4] v[5] v[6] + 7.34206 × 10^-10 v[5] v[6]^2 - 0.234738 v[7] +
  0.468249 v[1] v[7] - 0.233486 v[1]^2 v[7] + 7.04974 × 10^-6 v[2] v[7] -
  7.93465 × 10^-6 v[1] v[2] v[7] + 3.68316 × 10^-8 v[2]^2 v[7] - 0.179907 v[3] v[7] +
  0.179412 v[1] v[3] v[7] + 4.27332 × 10^-6 v[2] v[3] v[7] - 0.0344648 v[3]^2 v[7] +
  0.015515 v[4] v[7] - 0.0154711 v[1] v[4] v[7] - 4.40464 × 10^-7 v[2] v[4] v[7] +
  0.00594196 v[3] v[4] v[7] - 0.000255925 v[4]^2 v[7] + 0.0000197528 v[6] v[7] -
  0.0000216223 v[1] v[6] v[7] - 9.2874 × 10^-10 v[2] v[6] v[7] +
  8.59837 × 10^-6 v[3] v[6] v[7] - 7.64307 × 10^-7 v[4] v[6] v[7] - 2.45903 × 10^-10 v[6]^2 v[7]}
```

In[•]:=
```
$Path
```

Out[•]=
```
{C:\Users\Dror\AppData\Roaming\Mathematica\DocumentationIndices,
 C:\Program Files\Wolfram Research\Mathematica\11.3\SystemFiles\Links,
 C:\Users\Dror\AppData\Roaming\Mathematica\Kernel,
 C:\Users\Dror\AppData\Roaming\Mathematica\Autoload,
 C:\Users\Dror\AppData\Roaming\Mathematica\Applications,
 C:\ProgramData\Mathematica\Kernel, C:\ProgramData\Mathematica\Autoload,
 C:\ProgramData\Mathematica\Applications, ., C:\Users\Dror,
 C:\Program Files\Wolfram Research\Mathematica\11.3\AddOns\Packages,
 C:\Program Files\Wolfram Research\Mathematica\11.3\SystemFiles\Autoload,
 C:\Program Files\Wolfram Research\Mathematica\11.3\AddOns\Autoload,
 C:\Program Files\Wolfram Research\Mathematica\11.3\AddOns\Applications,
 C:\Program Files\Wolfram Research\Mathematica\11.3\AddOns\ExtraPackages,
 C:\Program Files\Wolfram Research\Mathematica\11.3\SystemFiles\Kernel\Packages,
 C:\Program Files\Wolfram Research\Mathematica\11.3\Documentation\English\System,
 C:\Program Files\Wolfram Research\Mathematica\11.3\SystemFiles\Data\ICC,
 C:\drorbn\projects\KnotTheory\svn\trunk, C:\drorbn\AcademicPensieve\Projects\UEA,
 C:\drorbn\projects\KnotTheory\svn\trunk\KnotTheory\WikiLink\mathematica\,
 C:\drorbn\projects\KnotTheory\svn\trunk\KnotTheory\QuantumGroups\}
```